Stopping and Choosing

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Topics in Economic Theory

Overview

Overview

Job Search

Accept offer Y_t, continue searching with a per period cost of c.

Interpretation:

- Job search (McCall 1970 QJE): TIOLI salary offers Y_t, cost to search c.
- Selling a house/asset: TIOLI offers Y_t, council tax/management fees c.

$Y_t \sim F$, iid; F continuous, strictly increasing.

Assume $\mathbb{E}[1_{Y_t \geq 0} Y_t] < \infty$; $Y_0 = 0$; $\mathbb{P}(Y_t > c) > 0$.

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    Refuse and get z and face same problem tomorrow

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Set up Bellman equation; $V(Y_t) = \max\{Y_t, \mathbb{E}[V(Y_{t+1})] - c\}$
(iid $\implies$ stationary problem)

Value: $V(Y_t)$
(handwavy: this presumes a solution and we don't know yet if/why we can do this)

Define $V_t := V(Y_t)$ and $\bar{V} = \mathbb{E}[V(Y_t)]$

Define $\tilde{y} := \bar{V} - c$

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Define $\tilde{y} := \bar{V} - c$

Take expectations and get $\tilde{y} + c = \mathbb{E}[\max\{Y_t, \tilde{y}\}] \iff c = \mathbb{E}[(Y_t - \tilde{y})^+] = \int_{\tilde{y}}^{\infty} y \, \mathrm{d}F(y)$

F continuous and strictly increasing: $\exists! \tilde{y} : c = \mathbb{E}[(Y_t - \tilde{y})^+]$

$\tilde{y}$: reservation value

Optimal rule: continue if and only if $Y_t < \tilde{y}$

Job Search with Discounting

Accept offer Y_t, continue searching and receive z; discount factor $\beta \in (0, 1)$.

Interpretation:

- Job search: TIOLI salary offers Y_t, unemployment subsidy z, cost of time β.
- Selling a house/asset: TIOLI offers Y_t, rent acrued z, interest rate r, discount factor $\beta = (1 + r)^{-1}$.

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Assume $\mathbb{E}[\mathbf{1}_{Y_t \geq 0} Y_t] < \infty$; $Y_0 = 0$; $\mathbb{P}(Y_t > c) > 0$.

Job Search with Discounting

Define $\hat{Y}_t := \frac{\beta^t}{1-\beta} Y_t$ (present value).

Accept and get Y_t forever $\equiv$ Accept and get $\hat{Y}_t$

Refuse, get z, and face same problem tomorrow but discounted by β.

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Brief refresher...

Refresher

Definition

$T : X \to X$ is a contraction on (X, d) if $\exists \delta \in [0, 1) : d(T(x), T(y)) \leq \delta d(x, y)\ \forall x, y \in X$.

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Banach Fixed-Point Theorem

Let (X, d) be a non-empty complete metric space and T a contraction mapping on (X, d). Then, $\exists! x^* \in X : T(x^*) = x^*$. Moreover, for any $x_0 \in X$, $x^* = \lim_{n\to\infty} T^n(x_0)$, where $T^{n+1} := T \circ T^n$ and $T^1 := T$.

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Proof

Let $x_n := T^n(x_0)$. Then $d(x_{n+1}, x_n) = d(T^n(x_1), T^n(x_0)) \leq \delta^n d(x_1, x_0)$, hence $\{x_n\}_n$ is a Cauchy sequence.

(X, d) complete $\equiv$ Cauchy sequences converge $\implies$ x_n converges to some $x^* = T(x^*)$.

Take any $y_0 \in X \setminus \{x_0\}$; define $y_n := T^n(y_0)$; $y_n \to y^*$.

If $x^* \neq y^*$, then $d(y^*, x^*) = d(T^n(y^*), T^n(x^*)) = \delta^n d(y^*, x^*) < d(y^*, x^*)$, a contradiction.

Refresher

Blackwell's Conditions for Contraction Mapping

Let $B(X)$ denote the set of bounded real functions on some nonempty set X endowed with the sup-metric d_∞. Suppose $T : B(X) \to B(X)$ satisfies (i) $\forall f, g \in B(X) : f \geq g \implies T(f) \geq T(g)$, and (ii) $\exists \delta \in [0,1)$ s.t. $T(f + \alpha) \leq T(f) + \delta\alpha \; \forall f \in B(X)$ and $\forall \alpha \in \mathbb{R}_+$. Then T is a contraction.

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Proof

For any $f, g \in B(X)$ and $x \in X$, $f(x) - g(x) \leq |f(x) - g(x)| \leq d_\infty(f, g)$.
(i) and (ii): $f \leq g + d_\infty(f, g) \implies T(f) \leq T(g) + \delta d_\infty(f, g)$
and, symmetrically, $T(g) \leq T(f) + \delta d_\infty(f, g)$.
This implies $d_\infty(T(f), T(g)) \leq \delta d_\infty(f, g)$.

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Markov problem; state variable = $\hat{Y}_t$

Set up Bellman equation; $V(\hat{Y}_t) = \max\{\hat{Y}_t, z + \beta\mathbb{E}[V(\hat{Y}_{t+1})]\}$

Value: $V(\hat{Y}_t)$, well-defined

Define $V_t := V(\hat{Y}_t)$ and $\bar{V} = \mathbb{E}[V(\hat{Y}_t)]$

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Define $V_t := V(\hat{Y}_t)$ and $\bar{V} = \mathbb{E}[V(\hat{Y}_t)]$

Take expectations and get

$$\bar{V} = \mathbb{E}[\max\{\hat{Y}_t, z + \beta\bar{V}\}] \iff \bar{V}(1-\beta) = z + \mathbb{E}[(\hat{Y}_t - (z + \beta\bar{V}))^+] = \int_{z+\beta\bar{V}}^{\infty} \frac{1}{1-\beta} y \, \mathrm{d}F(y)$$

F continuous: $\exists! \bar{V} : \bar{V}(1-\beta) = z + \mathbb{E}[(\hat{Y}_t - (z + \beta\bar{V}))^+]$

$\tilde{y} := (1-\beta)(z + \beta\bar{V})$: reservation value

Optimal rule: continue if and only if $Y_t < \tilde{y}$

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Going Beyond the Basic Setting

Y_t may not be iid

- Depend on time of unemployment
- Result from underlying dynamic game between recruiting firms
- Uncertain market conditions (hence perception of F evolves over time depending on past Y_ℓ)

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Introduce general tools to tackle the problem

Setup and Assumptions

$\{X_0, X_1, X_2, \ldots\}$ rv whose joint distribution is assumed to be known; write $X^t := (X_\ell)_{\ell=1,\ldots,t}$.

Sequence of functions $x^t \mapsto y_t(x^t) \in \mathbb{R}$; write $Y_t := y_t(x^t)$.

Filtration $\mathbb{F} = \{\mathcal{F}_t\} = \sigma(X^t)$.

Adapted payoff process $\{Y_t\}$; terminal Y_∞ (possibly $-\infty$).

Stopping time τ: $\{\tau \leq t\} \in \mathcal{F}_t$ for all t; feasible set $\mathbb{T}$.

Objective: maximise value of Y by adequately choosing stopping time, $\sup\limits_{\tau \in \mathbb{T}} \mathbb{E}[Y_\tau]$.

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Two questions:

1. When is there actually an optimal stopping time? (Is sup actually a max?)
2. If so, what does it look like?

   Previous applications: guess and verify or use specific structural assumptions.
   Now: use very general assumptions.

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Standing assumptions

(A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$.

(A2) $\lim_{t\to\infty} \mathbb{E}[Y_t] \leq Y_\infty$ a.s.

Note: (A1) implies $\sup_\tau \mathbb{E}[Y_\tau] < \infty$

Regular Stopping Times

Definition (Regularity)

τ is regular if for all t, $\mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. on $\{\tau > t\}$.

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Under (A1), for any stopping time τ there exists a *regular* stopping time $\rho \leq \tau$ with $\mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_\tau]$.

Lemma 2 (Regularity is closed under $\vee$)

Under (A1), if τ and ρ are regular, then $\xi := \tau \vee \rho$ is regular and $\mathbb{E}[Y_\xi] \geq \max\{\mathbb{E}[Y_\tau], \mathbb{E}[Y_\rho]\}$.

Proof of Lemma 1 (Regularity wloo)

Proof

Fix τ with $\mathbb{E}[|Y_\tau|] < \infty$ (true by (A1) since $Y_\tau \leq \sup_s Y_s$).

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Define $Z_t := \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$ and let $\rho := \inf\{t \ge \mathbf{0} : Z_t \le Y_t\}$.

On $\{\rho > t\}$: $Y_t < Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$, so ρ is regular.

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On $\{\rho = t\}$: $Y_\rho = Y_t \geq Z_t = \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$. On $\{\rho = \infty\}$: $Y_\rho = Y_\infty = Y_\tau$ a.s.

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Hence

$$\mathbb{E}[Y_\rho] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_t] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\infty]$$

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Hence

$$\begin{aligned}\mathbb{E}[Y_\rho] &= \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_t] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\infty] \\ &\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau]\end{aligned}$$

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&\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\rho=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\rho=\infty\}} Y_\tau] \\
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&= \mathbb{E}[Y_\tau].
\end{aligned}$$

Suppose $\neg(\rho \leq \tau)$; note that, at $\{\rho > \tau = t\}$, $Z_t = Z_\tau = Y_\tau < Z_t$, a contradiction.

Proof of Lemma 2 (Regularity is closed under $\vee$)

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1. Proving ξ is regular:

   $\{\xi > t\} = \{\xi = \tau > t\} \cup \{\xi = \rho > t\}$.

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   On $\{\xi = \tau > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\tau \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ τ is regular.

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   Symmetrically, on $\{\xi = \rho > t\}$, $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t$ a.s. $\because$ ρ is regular.

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   On $\{\xi = \tau = t\}$, $Y_\xi = Y_\tau = Y_t$.

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   On $\{\xi = \tau = t\}$, $Y_\xi = Y_\tau = Y_t$.

   On $\{\xi = \rho > \tau = t\}$, $\xi = \rho$ and $\mathbb{E}[Y_\xi \mid \mathcal{F}_t] = \mathbb{E}[Y_\rho \mid \mathcal{F}_t] > Y_t = Y_\tau$ a.s.

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   Hence

$$\begin{aligned}
\mathbb{E}[Y_\xi] &= \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\xi] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} \mathbb{E}[Y_\xi \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi] \\
&\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\tau] = \mathbb{E}[Y_\tau].
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   Hence

$$\mathbb{E}[Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\xi] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi] = \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} \mathbb{E}[Y_\xi \mid \mathcal{F}_t]] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\xi]$$

$$\geq \sum_{t=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{\tau=t\}} Y_\tau] + \mathbb{E}[\mathbf{1}_{\{\tau=\infty\}} Y_\tau] = \mathbb{E}[Y_\tau].$$

   By a symmetric argument, $\mathbb{E}[Y_\xi] \geq \max\{\mathbb{E}[Y_\tau], \mathbb{E}[Y_\rho]\}$.

Existence

Theorem (Existence)

Under (A1) and (A2), there is a regular τ such that $\mathbb{E}[Y_\tau] = \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho]$.

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Take the case $V^* := \sup_{\rho \in \mathbb{T}} \mathbb{E}[Y_\rho] > -\infty$.

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Define $\tau_\infty := \sup_{n \in \mathbb{N}} \rho_n$. τ_n pointwise increasing $\implies$ pointwise converges to τ_∞.
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Conclude: $V^* = \sup_{\rho\in\mathbb{T}} \mathbb{E}[Y_\rho] \geq \mathbb{E}[Y_{\tau_\infty}] \geq V^*$.

Assumptions

Example

Let X_t ~Bernoulli(1/2) iid; $Y_0 := \mathbf{0}$, $Y_t := (\mathbf{2}^t - \mathbf{1}) \prod_{\ell=\mathbf{1}}^{t} X_\ell$ for $t \in \mathbb{N}$, $Y_\infty := \mathbf{0}$.

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Fails (A1): Note $\sup_{k\leq t} Y_k = 2^k - 1$ with probability $2^{-(k+1)}$ for $k = 0, 1, \ldots, t-1$ and with probability 2^{-t} for $k = t$. Hence $\mathbb{E}[\sup_t Y_t] = \sum_{t=0}^\infty (2^t - 1)2^{-(t+1)} = \infty$.

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Indeed, no optimal stopping time. Conditional on reaching t with $Y_t > 0 \iff \prod_{\ell=1}^t X_\ell = 1$, then don't want to stop: $Y_t = 2^t - 1 < 2^t - 1/2 = (1/2)(2^{t+1} - 1) = \mathbb{E}[Y_{t+1} | Y_t > 0]$.

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Definition

Let $(X_t)_{t \in T}$ be a collection of rv. Z rv is essential supremum of $(X_t)_{t \in T}$, $Z = \operatorname{ess\,sup}_{t \in T} X_t$, if (i) $\mathbb{P}(Z \geq X_t) = 1 \; \forall t \in T$ ('probabilistic upper bound'), and (ii) $\forall Z' : \mathbb{P}(Z' \geq X_t) = 1 \; \forall t \in T$, $\mathbb{P}(Z' \geq Z) = 1$ (smallest probabilistic upper bound).

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Let $(X_t)_{t\in T}$ be any collection of rv.
An essential supremum always exists.
Furthermore, $\exists$ a countable $C \subset T : \sup_{t\in C} X_t = \operatorname{ess\,sup}_{t\in T} X_t$.

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Example

Let $U \sim U(0, 1)$, $T = [0, 1]$, and $X_t = 1_{\{U=t\}}$. $\sup_{t \in T} X_t = 1 \neq \operatorname{ess\,sup}_{t \in T} X_t = 0$.

Regularity from T Onward

Notation:

"$X \geq Y$" $\equiv \mathbb{P}(X \geq Y) = 1$.

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Lemma 2' (Regularity is closed under $\vee$)

Under (A1), if $\tau \geq T$ and $\rho \geq T$ are regular from T onward, then $\xi := \tau \vee \rho$ is regular from T onward and $\mathbb{E}[Y_\xi] \geq \max\{\mathbb{E}[Y_\tau], \mathbb{E}[Y_\rho]\}$.

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$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$$

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   Since, trivially, $V_t^* \geq Y_t$, we get $V_t^* \geq \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$.

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$$V_t^* := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t] = \max\{Y_t, \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}; \qquad \tau^* := \inf\{t \geq 0 \mid Y_t = V_t^*\}$$

Example

Let $Y_0 := 0$, $Y_t := 1 - 1/t$ for $t \in \mathbb{N}$, $Y_\infty := 0$.

Satisfies (A1): $Y_t \leq 1$.

Fails (A2): $Y_t \to 1 > 0 = Y_\infty$.

Indeed, no optimal stopping time as $Y_t < Y_{t+1}$.

Note: $\tau^* = \infty$ and $Y_{\tau^*} = 0 < V_t^*$.

Characterising Optimal Stopping Time

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Take any stopping time τ. Under (A1), $\mathbb{E}[Y_{\tau \wedge \tau^*}] \geq \mathbb{E}[Y_\tau]$.

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Theorem (Optimal Stopping Time)

Under (A1), if an optimal stopping time exists, τ^* is optimal.

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Finally, by Lemma 2, $\tau'' \vee \tau^*$ must also be optimal. Note that $\tau'' \vee \tau^* = \tau^*$ by construction.

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It can be shown that τ^* is the **earliest optimal stopping time**, i.e., $\tau^* \leq \tau \; \forall$ optimal τ.
(Intuition: If $\tau = t < \tau^*$, then $Y_t < V_t^*$ and an improvement can be reached)

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Another stopping time: $\tau^{**} := \inf\{t \geq 0 \mid Y_t > \mathbb{E}[V_{t+1}^* \mid \mathcal{F}_t]\}$

It can be shown that τ^{**} is the **latest optimal stopping time**, i.e., $\tau \leq \tau^{**} \; \forall$ optimal τ.

Overview

Satisficing

Oftentimes DM don't consider all items (virtually impossible in online shopping...).

DM knows there is a large set of feasible items but doesn't quite known what they are.

Upon stopping their search, pick best item available.

When to stop searching?

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- DM faces a large choice set A with T items.
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- Prior about the value of each option $X_t \overset{iid}{\sim} F$, absolutely continuous, strictly increasing.

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Proposition

Let $M_t := \max_{s \leq t} X_s$ and $\bar{x} : c = \int_{\bar{x}}^{\infty} (X - \bar{x})\, \mathrm{d}F(X)$. Then, $\tau_T^* := \inf\{t \geq 0 \mid M_t \geq \bar{x}\} \wedge T$ is optimal.

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Solving the Problem (Backwards induction intuition)

At $T-1$: stop and get $M_{T-1} - (T-1)c$ or continue and get $\mathbb{E}[M_T \mid \mathcal{F}_{T-1}] - Tc$.

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Stop at $T-1$ if $M_{T-1} \geq \bar{x}$.

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At $T - 2$**:** stop and get $M_{T-2} - (T-2)c$ or continue and get
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Suppose $M_{T-2} \ge \bar{x}$. Then upon continuing would stop at $T-1$ and get $\max\{M_{T-2}, X_{T-1}\} - (T-1)c$.

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Induction: $\tau^* := \inf\{t \geq 0 \mid M_t \geq \bar{x}\} \wedge T$ is optimal. (handwavy; rigorous proof later.)

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Satisficing solution: DM stops whenever has seen something “good enough”

$$\tau^*_T = \inf\{t \geq 0 \mid M_t \geq \bar{x}\} \wedge T.$$

Remark

$\tau^*_T = t < T \implies M_{\tau^*_T} = X_t.$

Stop only if last item seen is best! $\tau^*_T = t < T \implies M_{\tau^*_T} = X_t.$

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DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

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Then, $\bar{x} > \max_{s \neq t} X_s$ and DM won't stop before t because $M_{t-1} \leq \max_{s \neq t} X_s < \bar{x}$.

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Then, $\bar{x} > \max_{s \neq t} X_s$ and DM won't stop before t because $M_{t-1} \leq \max_{s \neq t} X_s < \bar{x}$.

At t, the DM stops ($X_t \geq \bar{x}$) and chooses t ($X_t > M_{t-1}$).

Satisficing

Proposition

DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

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Suppose $X_t \wedge \bar{x} < \max_{s \neq t} X_s \wedge \bar{x}$. Then X_t is never chosen.

Satisficing

Satisficing solution: DM stops whenever has seen something "good enough"

$\tau^*_T = \inf\{t \geq \mathbf{0} \mid M_t \geq \bar{x}\} \wedge T$.

Remark

$\tau^*_T = t < T \implies M_{\tau^*_T} = X_t$.

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DM chooses X_t if $X_t \wedge \bar{x} > \max_{s \neq t} X_s \wedge \bar{x}$ and only if $X_t \wedge \bar{x} \geq \max_{s \neq t} X_s \wedge \bar{x}$.

Corollary

$\mathbb{E}[X_{\tau^*_T}] = \mathbb{E}[\max_{t \leq T} X_t \wedge \bar{x}]$.

Dependence on c only through $\bar{x}$.

Satisficing

Remark

$\mathbb{E}[\tau_T^*] = \frac{1-F(\bar{x})^{T-1}}{1-F(\bar{x})}$.

Proof

Since $\mathbb{P}(\tau_T^* \geq t) = 1-\mathbb{P}(\tau_T^* \leq t-1) = 1-1_{\{t\leq T\}}\sum_{s=1}^{t-1}(1-F(\bar{x}))F(\bar{x})^{s-1} = 1-1_{\{t\leq T\}}(1-F(\bar{x})^{t-1})$.

Then, $\mathbb{E}[\tau_T^*] = \sum_{t=1}^{T}\mathbb{P}(\tau_T^* \geq t) = \frac{1-F(\bar{x})^{T-1}}{1-F(\bar{x})}$.

Note that

$\text{sign}(\frac{\partial}{\partial \bar{x}}\mathbb{E}[\tau_T^*]) = \text{sign}(F(\bar{x})+F(\bar{x})^{T+1}(T-1)-F(\bar{x})^{T}T) = \text{sign}(1+F(\bar{x})^{T}(T-1)-F(\bar{x})^{T-1}T) > 0$ for $F(\bar{x}) \in (0,1)$.

Comparative Statics

Remark

(i) $\uparrow c \implies \downarrow \bar{x} \implies \downarrow \mathbb{E}[X_{\tau_T^*}], \mathbb{E}[\tau_T^*]$;

(ii) F' MPS of $F \implies \bar{x}' \geq \bar{x}$ (higher option value) $\implies \uparrow \mathbb{E}[X_{\tau_T^*}], \mathbb{E}[\tau_T^*]$;

(iii) $F'(x) = F(x - \mu)$ (shift in mean) $\implies \bar{x}' = \bar{x} + \mu$

$\implies \mathbb{E}[X'_{\tau_T^*}] = \mathbb{E}[X_{\tau_T^*}] + \mu, \quad \mathbb{E}[\tau_T^*] = \mathbb{E}[\tau_T^{*\prime}]$;

(iv) $\bar{x}$ remains constant wrt T.

Overview

Setup and Assumptions

$\{X_0, X_1, X_2, ...\}$ rv whose joint distribution is assumed to be known; write $X^t := (X_\ell)_{\ell=1,...,t}$.

Sequence of functions $x^t \mapsto y_t(x^t) \in \mathbb{R}$; write $Y_t := y_t(x^t)$.

Filtration $\mathbb{F} = \{\mathcal{F}_t\} = \sigma(X^t)$.

Adapted payoff process $\{Y_t\}$; terminal Y_∞ (possibly $-\infty$).

Stopping time τ: $\{\tau \leq t\} \in \mathcal{F}_t$ for all t; feasible set $\mathbb{T}$.

Truncation: T terminal time; Stopping times: $\mathbb{T}_T = \{\tau \leq T\}$.

Objective: maximise value of Y by adequately choosing stopping time,

$$V_0^{(T)} := \sup_{\tau \in \mathbb{T}_T} \mathbb{E}[Y_\tau].$$

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Backward Induction: $V_T^{(T)} := Y_T$; $V_t^{(T)} := \max\{Y_t, \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, $t = T-1, T-2, ..., 0$.

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For truncation in problems when continuing forever is valuable, replace

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Simple Stopping Rules

One-Stage Look-Ahead Stopping Time: $\tau_{1\text{-sla}} := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\}$.

Myopic stopping: stop as soon as it doesn't pay off continuing for one more period.

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Stop if continuing for at most k more periods isn't worthwhile.

Naive: not considering that next period will consider the following k periods instead of $k-1$.

Note: $Y_t \geq \mathbb{E}[V_{t+1}^{(t+k)} \mid \mathcal{F}_t] \iff Y_t \geq V_t^{(t+k)} \because V_t^{(t+k)} = \max\{Y_t, \mathbb{E}[V_{t+1}^{(t+k)} \mid \mathcal{F}_t]\}$.

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One-Time Look-Ahead Stopping Time: $\tau_{1\text{-tla}} := \inf\{t \geq 0 \mid Y_t \geq \sup_{\ell>0} \mathbb{E}[Y_{t+\ell} \mid \mathcal{F}_t]\}$.

Continue iff $\exists \ell > 0$: committing to continue ℓ periods more is better than stopping.

Naively committed: $t + \ell$ may decide to continue again.

$\tau_{1\text{-sla}} \leq \tau_{1\text{-tla}}, \tau_{k\text{-sla}} \leq \tau^*$.

Moreover, $\mathbb{E}[Y_{\tau_{1\text{-sla}}}] \leq \mathbb{E}[Y_{\tau_{k\text{-sla}}}], \mathbb{E}[Y_{\tau_{1\text{-sla}}}] \leq \mathbb{E}[Y_{\tau^*}]$.

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Let horizon be T. Earliest optimal stopping $\tau^* := \inf\{t \geq 0 \mid Y_t \geq \mathbb{E}[V_{t+1}^{(T)} \mid \mathcal{F}_t]\}$, with $V_{T+1}^{(T)} = -\infty$ and $V_T^{(T)} = Y_T$.

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Fix $t < T$. Note $\tau_{\text{1-sla}} > t \implies \tau^* > t$. Suppose $\tau_{\text{1-sla}} = t$.

Since $\{\tau_{\text{1-sla}} = t\} = \{Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\} = A_t$ and problem is monotone,

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Hence, $\tau^* = t$.

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Definition

$\{X_t\}$ are uniformly integrable if $\lim_{a\to\infty} \sup_t \mathbb{E}[|X_t|\mathbf{1}_{\{|X_t|>a\}}] = 0$.

Conditions for uniform integrability:

1. $\lim_{t\to\infty} \mathbb{E}[|X_t|] = 0$, then $\{X_t\}_t$ is uniform integrable.
2. $\lim_{t\to\infty} \mathbb{E}[|X_t|] = \infty$, then $\{X_t\}_t$ is not uniform integrable.

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Let $p_T = \mathbb{P}(Y_\infty - Y_T > q_T)^{-1/2} \to \infty$.

By uniform integrability of Z_T, $\mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] = \mathbb{E}[\mathbf{1}_{\{A_T\}}\mathbf{1}_{\{Z_T \leq q_T\}} Z_T] + \mathbb{E}[\mathbf{1}_{\{A_T\}}\mathbf{1}_{\{Z_T > q_T\}} Z_T] \leq q_T \mathbb{P}(A_T) + \mathbb{E}[\mathbf{1}_{\{Z_T > q_T\}} Z_T] \to 0$.

Approximating Infinite Horizon by Finite Horizon

Standing assumptions: (A1) $\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] < \infty$. (A3) $\lim_{t\to\infty} Y_t = Y_\infty$ a.s.

Theorem

Assume (A1) and (A3). If $Z_t := \sup_{j\geq t} Y_j - Y_t$ is uniformly integrable, then $V_{\mathbf{0}}^{(T)} \to V^*$ as $T \to \infty$.

Proof

Let $A_T := \{T < \tau^* < \infty\}$. $\quad \mathbf{0} \leq V^* - V_{\mathbf{0}}^{(T)} \leq \mathbb{E}[\mathbf{1}_{\{A_T\}} Z_T] + \mathbb{E}[(Y_\infty - Y_T)^+]$.

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$\mathbb{E}[(Y_\infty - Y_T)^+] = \mathbb{E}[\mathbf{1}_{\{Y_\infty - Y_t \leq \varepsilon_T\}}(Y_\infty - Y_T)^+] + \mathbb{E}[\mathbf{1}_{\{Y_\infty - Y_T > \varepsilon_T\}}(Y_\infty - Y_T)^+] \leq \varepsilon_T + \mathbb{E}[\mathbf{1}_{\{Y_\infty - Y_T > \varepsilon_T\}} Z_T]$.

$\mathbb{E}[\mathbf{1}_{\{Y_\infty - Y_T > \varepsilon_T\}} Z_T] \to \mathbf{0}$ follows by similar argument as before for 1st term.

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Corollary

Assume (A3). If $Y_t := B_t - C_t$, where $\mathbb{E}[\sup_t |B_t|] < \infty$ and $C_t \geq 0$ and nondecreasing a.s., then (A1) holds and $V_0^{(T)} \to V^*$.

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Proof

$\mathbb{E}\left[\sup_{t\geq 0} Y_t\right] \leq \mathbb{E}\left[\sup_{t\geq 0} |B_t|\right] < \infty \implies$ (A1) holds.

For $j \geq t$, $Y_j - Y_t = B_j - B_t + (C_t - C_j) \leq B_j - B_t$.

$0 \leq Z_t := \sup_{j\geq t} Y_j - Y_t \leq 2 \sup_t |B_t| =: B'$.

$\mathbb{E}[B'] < \infty$, hence $\mathbb{E}[1_{\{|Z_t|>a\}}|Z_t|] \leq \mathbb{E}[1_{\{B'>a\}}B'] \to 0$ and Z_t is uniformly integrable.

Overview

Stopping and Choosing: Selling a House

Accept best offer M_t or continue waiting with a per period cost of c.

Interpretation:

- Selling a house/asset: offers $X_t \geq 0$ come in, council tax/management fees c;
  - $Y_t := M_t - ct$, where $M_t := \max_{s \leq t} X_t$.
- Same as satisficing, just take $T = \infty$.

$X_t \sim F$, iid; F continuous, strictly increasing, with finite 2nd moment.

Stopping and Choosing: Selling a House

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Set up Bellman equation; $V(Y_t) = \max\{Y_t, \mathbb{E}[V(Y_{t+1})] - c\}$.

Define $V_t := V(Y_t)$; $\mathbb{E}[V(Y_t)]$ now depends on t!

Simple derivation from before no longer works...

Stopping and Choosing: Selling a House

- Accept and get $M_t - tc$;
  - Refuse and pay c and wait for one more offer tomorrow.
- Markov problem; state variable = Y_t
- Set up Bellman equation; $V(Y_t) = \max\{Y_t, \mathbb{E}[V(Y_{t+1})] - c\}$.
- Define $V_t := V(Y_t)$; $\mathbb{E}[V(Y_t)]$ now depends on t!
  - Simple derivation from before no longer works...
- But this is a **monotone problem**:
  - $Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t] \iff Y_t \geq \mathbb{E}[\max\{Y_t, X_{t+1} - tc\} \mid \mathcal{F}_t] - c \iff c \geq \mathbb{E}[(X_0 - (Y_t + tc))^+ \mid \mathcal{F}_t]$.
  - Since $Y_t + tc$ is increasing in t, $\{Y_t \geq \mathbb{E}[Y_{t+1} \mid \mathcal{F}_t]\} \subseteq \{Y_{t+\ell} \geq \mathbb{E}[Y_{t+\ell+1} \mid \mathcal{F}_{t+\ell}]\}$ for any $t \geq 0$ and $\ell \geq 0$.
- Check conditions for approximation: (A1), (A3), and UI...

Stopping and Choosing: Selling a House

Theorem

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If $\mathbb{E}[X^+] < \infty$, then $\sup_t Y_t < \infty$ a.s. and $Y_t \to -\infty$ a.s.

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See the proof to Theorem 1 in Ferguson (2008, Ch. 4, Appendix).

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Conclude 1-sla is still optimal with infinite horizon!

Stopping and Choosing: Selling a House

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$Y_t := X_t - tc, X_t \sim F$ iid.

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Stopping and Choosing: Selling a House

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This is *not* a monotone problem!

Selling a house with distributional uncertainty:

$Y_t := M_t - tc$, $X_t \sim F(\cdot \mid \theta)$ iid, but θ unknown, $\theta \sim P$.

Let $\mathbb{E}[\mathbf{1}_{\{X_t \leq \cdot\}} \mid \mathcal{F}_t] = F_t$ and suppose that $F_t = \frac{\alpha_0}{\alpha_0 + t} F_0 + \frac{t}{\alpha_0 + t} \hat{F}_t$, where $\hat{F}_t$ is ECDF, $\alpha_0 > 0$, and F_0 has finite 2nd moment. (E.g., Dirichlet process prior.)

This is a monotone problem and 1-sla is still optimal. Prove it!

Overview

The Diamond (1971 JPE) Model (adapted)

Foundational model of price search.

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Environment

- N identical sellers; homogenous good; zero marginal cost (normalisation).
- Identical mass 1 of consumers; unit demand (generalises).
- Known valuation $v > \mathbf{0}$. Value from purchase at price $\hat{p}$ is $v - \hat{p}$.

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Timing

Sellers set prices $p = \{p^n\} \subset \mathbb{R}_+$.

Consumer knows empirical distribution of prices,
but not which seller sets which price.

Consumer learns price of seller n only by visiting seller.
Visit bears a cost $c > \mathbf{0}$. (visit, browse, ask for a quote, etc.)

Sellers selected to visit uniformly at random (among those not yet visited).

Following each visit, consumer can either choose to buy good from one of the sellers they visited or to learn the price of another seller.

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Key Features

Uncertainty over *prices*, not match values.

Solving for Equilibrium Prices

Notation

$n_t \in \{1, ..., N\}$: seller sampled at t.

$S_t := \{n_1, ..., n_t\}$: sellers sampled by t (consideration set).

$N_t := \{1, ..., N\} \setminus S_t$: sellers not yet sampled by t.

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$X_t := v - p_t$; $\quad M_t := \max_{s \le t} v - p_s$; $\quad Y_t := M_t - tc$; $\quad V_t := \operatorname{ess\,sup}_{\tau \ge t} \mathbb{E}[Y_\tau \mid \mathcal{F}_t]$.

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Fix prices and label sellers: $\underline{p} = p^1 \leq \cdots \leq p^N = \overline{p}$.

τ: optimal stopping by consumer.

Note: $Y_t = v - \underline{p} - tc \implies \tau \leq t$.

Claim

$\underline{p} = \overline{p} \leq v$.

Solving for Equilibrium Prices

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Proof

Suppose not. If $\overline{p} > v$, then seller N has strict incentive to lower price to $v - \varepsilon$ for some small enough $\varepsilon > 0$. Then $\underline{p} = p^1 < p^N = \overline{p} \leq v$.

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WTS that seller 1 can increase profits by increasing the price.

Solving for Equilibrium Prices

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Prob. purchase 1 = $\mathbb{P}(n_{t+1} = \mathbf{1} \text{ and } \boldsymbol{\tau} > t)$.

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Prob. purchase 1 = $\mathbb{P}(n_{t+1} = \mathbf{1} \text{ and } \boldsymbol{\tau} > t)$.

Case 1. At $\{n_{t+1} = \mathbf{1}\} \cap \{\boldsymbol{\tau} \leq t\}$, seller 1 gets 0, so increasing p_1 does not lower profits.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} \leq v$.

Proof

Prob. purchase 1 = $\mathbb{P}(n_{t+1} = \mathbf{1} \text{ and } \tau > t)$.

Case 1. At $\{n_{t+1} = \mathbf{1}\} \cap \{\tau \leq t\}$, seller 1 gets 0, so increasing p_1 does not lower profits.

Case 2. At $\{n_1 = \mathbf{1}\}$, $\tau = \mathbf{1}$. Seller 1 can increase p_1 by $c/\mathbf{2}$ and still deter further search: continuation value is at best $v - \underline{p} - \mathbf{2}c < v - (\underline{p} + c/\mathbf{2}) - c$ = value of stopping and paying $\underline{p} + c/\mathbf{2}$.

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Prob. purchase 1 = $\mathbb{P}(n_{t+1} = 1 \text{ and } \tau > t)$.

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Case 2. At $\{n_1 = 1\}$, $\tau = 1$. Seller 1 can increase p_1 by $c/2$ and still deter further search: continuation value is at best $v - \underline{p} - 2c < v - (\underline{p} + c/2) - c$ = value of stopping and paying $\underline{p} + c/2$.

Case 3. At $\{n_{t+1} = 1\} \cap \{\tau > t\}$ $t \geq 1$, it must be that $M_t < v - \underline{p}$.

$\tau > t \implies \mathbb{E}[V_{t+1} \mid \mathcal{F}_t] - Y_t =: \varepsilon(M_t; p) > 0$.

More: conditional on $\tau > t$, $\exists$ finitely many values possible for $M_t \in \hat{M} := \{v - \hat{p}, \hat{p} \in \{p^1, ..., p^N\} \setminus \{\underline{p}\}\}$.

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Hence, seller 1 can increase p_1 by $\varepsilon' = \min_{M \in \hat{M}} \varepsilon(M; p)/2$ and still deter further search.

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Hence, seller 1 can increase p_1 by $\varepsilon' = \min_{M \in \hat{M}} \varepsilon(M; p)/2$ and still deter further search.

Increasing price never reduces profits for seller 1 and as Case 2 occurs wp> 0: found strictly profitable deviation.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} = v$.

Proof

Suppose $\underline{p} = \overline{p} < v$. WTS that seller 1 can increase profits by increasing the price.

Purchase 1 only if $n_1 = \mathbf{1}$.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} = v$.

Proof

Suppose $\underline{p} = \overline{p} < v$. WTS that seller 1 can increase profits by increasing the price.

Purchase 1 only if $n_1 = \mathbf{1}$.

Case 1. At $\{n_1 \neq \mathbf{1}\}$, seller 1 gets zero, so increasing its price does not harm profits.

Solving for Equilibrium Prices

Claim

$\underline{p} = \overline{p} = v$.

Proof

Suppose $\underline{p} = \overline{p} < v$. WTS that seller 1 can increase profits by increasing the price.

Purchase 1 only if $n_1 = 1$.

Case 1. At $\{n_1 \neq 1\}$, seller 1 gets zero, so increasing its price does not harm profits.

Case 2. At $\{n_1 = 1\}$. Seller 1 can increase price by $c/2$ while still deterring further search: continuation value is $v - \underline{p} - 2c < v - (\underline{p} + c/2) - c$ = value of stopping and paying $\underline{p} + c/2$.

Solving for Equilibrium Prices

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$\underline{p} = \overline{p} = v$.

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Increasing price is strictly profitable deviation.

Diamond's Paradox

Implications

"Who cares about search costs in the digital age? Such costs are minute!"

Diamond's Paradox

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The Paradox

Any arbitrarily small search cost ($c > 0$) causes the market outcome to jump discontinuously from competitive Bertrand outcome ($p = 0$) to full monopoly outcome ($p = V$)! Slightest search friction destroys all price competition.

A Variation on Diamond's Paradox

Burdett & Judd (1983 Ecta): when sampling, instead of getting one price quote, get random sample of k price quotes.

If $\mathbb{P}(k = 1) = 1$, Diamond model; get monopoly price.

If $\mathbb{P}(k = 1) = 0$, Bertrand competition; get competitive price.

If $\mathbb{P}(k = 1) \in (0, 1)$ get price dispersion!

Overview

Some jargon:

With recall: possibility of choosing any of the samples thus far. *Without recall:* can only choose current element or sample again.

Without replacement: samples are all distinct. *With replacement:* can resample previously observed sample.

Undirected search: fixed order. *Directed search:* choose the order (more next lecture).

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Stopping and Choosing

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